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Set Packing Relaxations of Some Integer Programs

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Abstract. This paper is about *set packing relaxations* of combinatorial optimization problems associated with acyclic digraphs and linear orderings, cuts and multicuts, and vertex packings themselves. Families of inequalities that are valid for such a relaxation as well as the associated separation routines carry over to the problems under investigation.

Keywords. Set Packing, Polyhedral Combinatorics, Cutting Planes, Integer Programming

Mathematics Subject Classification (MSC 1991). 90C10

1 Introduction

This paper is about *relaxations* of certain combinatorial optimization problems in the form of a set packing problem and the use of such relaxations in a polyhedral approach.

The *set packing* or *stable set problem* (SSP) is to find, in a graph $G = (V, E)$ with node weights c, a *stable set*, i.e., a set of pairwise non-adjacent nodes, of maximum weight. Set packing problems are among the best studied combinatorial optimization problems with beautiful theories connecting this area of research to Fulkerson's anti-blocking theory, the theory of perfect graphs, perfect and balanced matrices, semidefinite programming, and other fields, see Grötschel, Lovász & Schrijver [1988] for a survey. Likewise, the *set packing polytope*, i.e., the convex hull of all node packings of a graph, plays a prominent rôle in polyhedral combinatorics not only because large classes of (facet defining) inequalities are known. Perhaps even more important, many of them can be separated in polynomial time. In particular, odd-cycle, orthogonality, and (superclasses of) anti-hole constraints are polynomial time separable, see again Grötschel, Lovász & Schrijver [1988] and Lovász & Schrijver [1991].

Our aim in this paper is to transfer some of these results to other combinatorial optimization problems. We show that the acyclic subdigraph and the linear ordering problem, the max-cut, the k-multicut, and the clique partitioning problem, and the set packing problem itself have interesting *combinatorial relaxations* in form of a set packing problem. Families of inequalities that are valid for these relaxations and the associated separation routines carry over to the problems under investigation. The procedure is an application of a more general method to construct relaxations of combinatorial optimization problems by means of *affine transformations* that we discuss in a forthcoming paper.

The paper is subdivided into three parts. Part 1 is devoted to a study of the acyclic subdigraph and the linear ordering problem, see Grötschel, Jünger & Reinelt [1985a,b]. A main result in this section is that a class of Möbius ladders with dicycles of *arbitrary* length belong to a (larger) class of odd cycles of an appropriate set packing relaxation; this superclass is polynomial time separable. Part 2 deals with set packing relaxations of the clique partitioning, the k-multicut, and the max-cut problem, see Grötschel & Wakabayashi [1990] and Deza & Laurent [1997]. We introduce two types of "inequalities from odd cycles of triangles". The first of these classes contains the 2-chorded cycle inequalities, the second is related to circulant inequalities. Part 3 treats the set packing problem *itself*. We show, in particular, that the wheel inequalities of Barahona & Mahjoub [1994] and Cheng & Cunningham [1997] are odd cycle inequalities of a suitable set packing relaxation. We also introduce a new family of facet-defining inequalities for the set packing polytope: The "cycle of cycles" inequalities. This class can be separated in polynomial time.

The subsequent sections resort to the following notation and results about the set packing problem. The set packing problem for a graph G with node weights c can be formulated as an *integer program*

$$\text{(SSP)} \qquad \begin{array}{rcl} \max c^T x & & \\ Ax & \leq & \mathbf{1} \\ x & \in & \{0,1\}^V, \end{array}$$

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where $A = A(G) \in \{0,1\}^{E \times V}$ is the edge-node incidence matrix of G and $\mathbf{1}$ a vector of all ones of compatible dimension. Without loss of generality, we can assume $c > 0$. Associated to this program is the *stable set polytope*

$$P_{\text{SSP}} := \text{conv}\left\{\chi^S : \; S \text{ is a stable set in } G\right\} = \text{conv}\left\{x \in \{0,1\}^n : \; Ax \leq \mathbf{1}\right\},$$

the convex hull of all incidence vectors of stable sets in G or, equivalently, of all solutions of (SSP). Occasionally, we will denote this polytope also by $P_{\text{SSP}}(G)$. For technical reasons, we will actually not work with the stable set polytope P_{SSP} itself, but with its *anti-dominant*

$$\check{P}_{\text{SSP}} := P_{\text{SSP}} - \mathbb{R}^V_+ = \left\{x \in \mathbb{R}^V : \exists y \in P_{\text{SSP}} : x \leq y\right\}.$$

This construction allows to consider vectors with arbitrary negative coordinates *without* destroying the polyhedral structure of P_{SSP}: Obviously, the valid inequalities for $\check{P}_{\text{SSP}}$ are exactly the valid inequalities for P_{SSP} *with non-negative coefficients.* Since the stable set polytope P_{SSP} is down-monotone, its non-trivial constraints all have non-negative coefficients, and we can thus work with $\check{P}_{\text{SSP}}$ as well as with P_{SSP}.

We will need a few results on P_{SSP} (or $\check{P}_{\text{SSP}}$) and quickly resume some terminology to state them. A *clique* in a graph G is a set of pairwise adjacent nodes, a *walk* is a sequence $v_1, e_1, v_2, e_2, \ldots, e_k, v_{k+1}$ of nodes v_i and edges e_i such that $e_i = v_i v_{i+1}$, a walk is a *path* if all its nodes are different, except possibly the first and the last, which are allowed to be identical; in this case, the path is called a *cycle.* The analogous concepts for digraphs, with the additional stipulation that all arcs have to be "oriented in the same direction", are *dipaths* and *dicycles.* A cycle is *odd* if k is odd, i.e., if it contains an odd number of nodes. An edge that joins two nodes of a cycle, but is not a member of the cycle, is called a *chord*, a chordless cycle is a *hole.* For convenience of notation, we will occasionally consider (di)paths and (di)cycles as *sets* of nodes, edges, or arcs, and we will denote edges *as well as* arcs with the symbols ij and (i,j); the latter will be used in cases like $(i, i+1)$.

The results on P_{SSP} that we need are summarized in the following two theorems; we state them for $\check{P}_{\text{SSP}}$.

1.1 Theorem (Clique and Odd Cycle Inequalities, Padberg [1973])
Let $G = (V,E)$ be a graph and $\check{P}_{\text{SSP}}$ the anti-dominant of the associated set packing polytope.

(i) If Q is a clique in G, the clique inequality *$\sum_{i \in Q} x_i \leq 1$ is valid for $\check{P}_{\text{SSP}}$; it is facet-defining if and only if Q is a maximal clique (with respect to set inclusion).*

(ii) If $C \subseteq V$ is the node set of an odd cycle in G, the odd cycle inequality *$\sum_{i \in C} x_i \leq (|C|-1)/2$ is valid for $\check{P}_{\text{SSP}}$.*

Separation of clique inequalities is $\mathcal{NP}$-hard. But the clique inequalities are contained in a class of *orthogonality constraints*, see Grötschel, Lovász & Schrijver [1988], that can be separated in polynomial time.

1.2 Theorem (Orthogonality and Cycle Inequalities, Grötschel, Lovász & Schrijver [1988])
Let $G = (V,E)$ be a graph, $\check{P}_{\text{SSP}}$ the anti-dominant of the associated set packing polytope, and $x \in \mathbb{Q}^V$. Suppose that $x_i + x_j \leq 1$ holds for all edges $ij \in E$. Then:

(i) Orthogonality constraints can be separated in polynomial time.

(ii) Odd cycle inequalities can be separated in polynomial time.

2 The Acyclic Subdigraph and the Linear Ordering Problem

Our aim in this section is to construct a set packing relaxation of the acyclic subdigraph and the linear ordering problem in a space of exponential dimension. It will turn out that clique and odd cycle inequalities of this relaxation give rise to (and generalize) several classes of inequalities for the acyclic subdigraph and the linear ordering problem, namely, fence and Möbius ladder inequalities. References are Grötschel, Jünger & Reinelt [1985a] for the acyclic subdigraph problem, and Grötschel, Jünger & Reinelt [1985b] for the linear ordering problem.

The acyclic subdigraph and the linear ordering problem involve a complete digraph $D_n = (V, A)$ on n nodes with integer weights w_a on its arcs $a \in A$. An *acyclic arc set* in A contains no dicycle. The *acyclic subdigraph problem* (ASP) asks for an acyclic arc set with maximum weight on its arcs. Acyclic arc sets that contain, for any pair of nodes i and j, either the arc ij or the arc ji, are called *tournaments*. The *linear ordering problem* (LOP) is to find a tournament of maximum weight. Integer programming formulations for the ASP and the LOP read as follows:

$$
\begin{array}{lrll}
\max & \sum_{ij \in A} w_{ij} x_{ij} & & \\
\text{(ii)} & \sum_{ij \in C} x_{ij} \le |C| - 1 & \forall \text{dicycles } C \subseteq A \\
\text{(iii)} & -x_{ij} \le 0 & \forall ij \in A \\
\text{(iv)} & x_{ij} \le 1 & \forall ij \in A \\
\text{(v)} & x_{ij} \in Z & \forall ij \in A
\end{array}
\qquad \text{(ASP)}
$$

$$
\begin{array}{lrll}
\max & \sum_{ij \in A} w_{ij} x_{ij} & & \\
\text{(i)} & x_{ij} + x_{ji} = 1 & \forall i, j \in V, i \neq j \\
\text{(ii)} & \sum_{ij \in C} x_{ij} \le |C| - 1 & \forall \text{dicycles } C \subseteq A \\
\text{(iii)} & -x_{ij} \le 0 & \forall ij \in A \\
\text{(iv)} & x_{ij} \le 1 & \forall ij \in A \\
\text{(v)} & x_{ij} \in Z & \forall ij \in A.
\end{array}
\qquad \text{(LOP)}
$$

(ASP) is a relaxation of (LOP) and, what is more, the *linear ordering polytope* P_{LOP} is a face of the *acyclic subdigraph polytope* P_{ASP}. In particular, all inequalities that are valid for P_{ASP} are also valid for P_{LOP}. Two such classes of inequalities for both the ASP and the LOP are the *k-fence* and the *Möbius ladder inequalities*, see Grötschel, Jünger & Reinelt [1985a].

A *k-fence* involves two disjoint sets of "upper" and "lower" nodes $\{u_1, \dots, u_k\}$ and $\{l_1, \dots, l_k\}$ that are joined by a set of k *pales* $\{u_1 l_1, \dots, u_k l_k\}$. All pales are oriented "downward". The k-fence is completed by adding all "upward" *pickets* $l_i u_j$ with the exception of the antiparallel pales. Figure 1 shows a 4-fence.

A *Möbius ladder* consists of an odd number $2k+1$ of dicycles $C_0, \dots, C_{2k}$ such that C_i and C_{i+1} (indices taken modulo $2k+1$) have a dipath P_i in common, see Figure 2.

Fences and Möbius ladders give rise to valid inequalities for P_{ASP}: For a k-fence F_k and a Möbius ladder M of $2k+1$ dicycles we have

$$\sum_{ij \in F_k} x_{ij} \le k^2 - k - 1 \qquad \text{and} \qquad \sum_{i=0}^{2k} \sum_{ij \in C_i \setminus P_i} x_{ij} \le \left(\sum_{i=0}^{2k} |C_i \setminus P_i| \right) - (k+1).$$

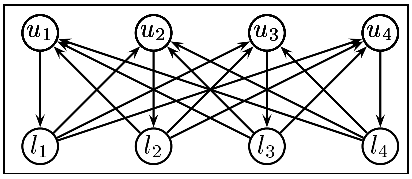


Figure 1: A 4-Fence.

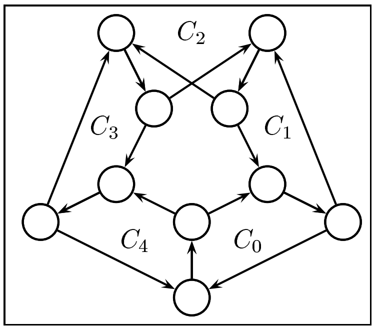


Figure 2: A Möbius Ladder of 5 Dicycles.

Note that a Möbius ladder inequality as above has coefficients larger than one if an arc is contained in more than one of the dipaths $C_i \setminus P_i$. This is different from Grötschel, Jünger & Reinelt [1985a]'s (original) definition, where the coefficients take only values of zero and one and the Möbius ladder must meet a number of additional technical requirements to support a valid inequality. The two definitions of a Möbius ladder inequality coincide if and only if no two dipaths $C_i \setminus P_i$ have an arc in common (Möbius ladder without *arc repetition*).

We will show now that fences and Möbius ladders are cliques and odd cycles, respectively, in an (exponential) *conflict graph* $\mathfrak{G}(D_n) = (\mathfrak{V}, \mathfrak{E})$. $\mathfrak{G}$ has the set of all acyclic arc sets of D_n as its nodes. We draw an edge $\mathfrak{u}\mathfrak{v}$ between two acyclic arc set-nodes $\mathfrak{u}$ and $\mathfrak{v}$, if their union contains a dicycle. In this case, we say that $\mathfrak{u}$ *and* $\mathfrak{v}$ *are in conflict*, because they can not be simultaneously contained in (the support of) a solution to (ASP).

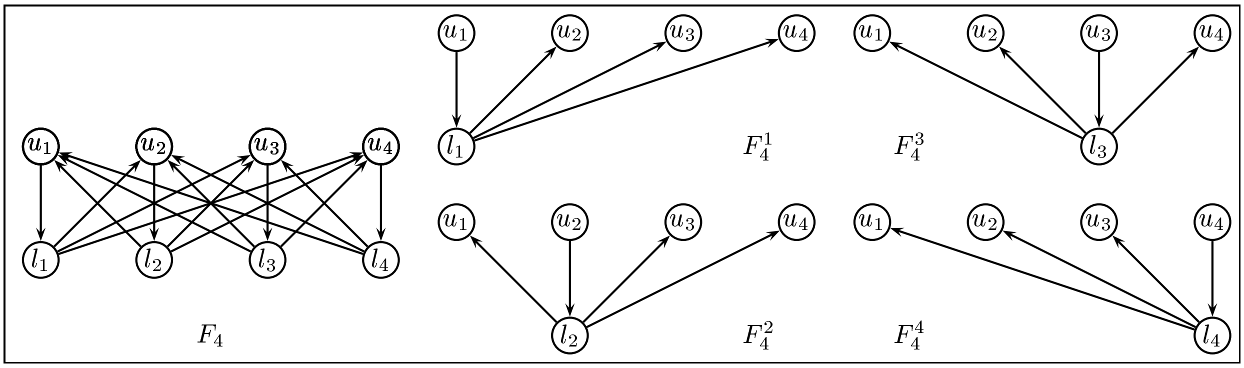


Figure 3: A Fence Clique.

It is now easy to identify the fences and Möbius ladders with cliques and odd cycles of $\mathfrak{G}$. To obtain a k-fence F_k, we look at the k acyclic arc sets F_k^i that consist of a pale u_il_i and the pickets l_iu_j that go up from l_i, for $i = 1, \dots, k$. Any two such configurations F_k^i and F_k^j, $i \neq j$, are in conflict (they contain a dicycle). Hence, all of them together form a clique. Figure 3 illustrates this construction. Likewise, the Möbius ladders correspond to odd cycles of conflicting dipaths, namely, the dipaths $C_i \setminus P_i$, see Figure 4.

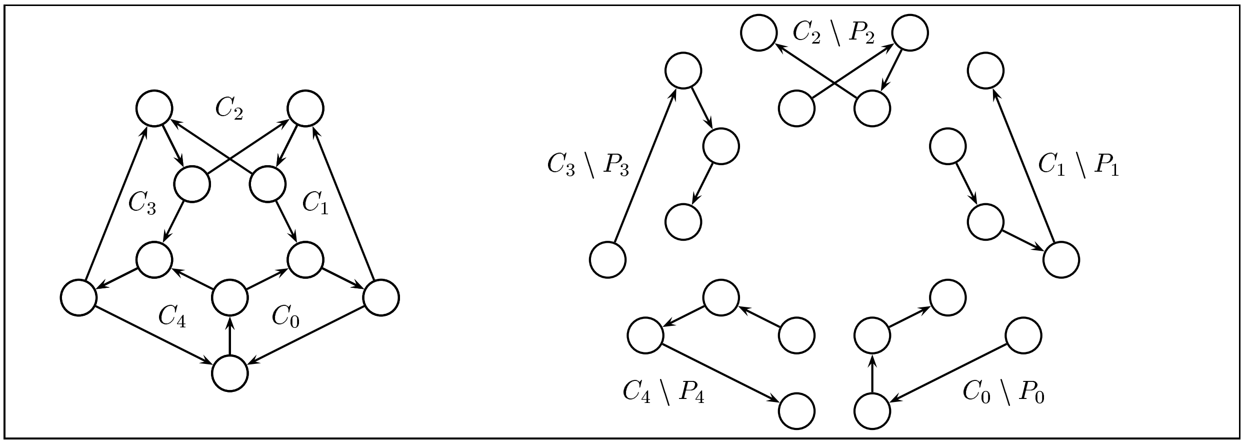


Figure 4: A Möbius Cycle of Dipaths.

The next step to obtain the fence and the Möbius ladder inequalities from the clique and odd cycle inequalities of the (anti-dominant of the) set packing polytope $\breve{P}_{\mathrm{SSP}}(\mathfrak{G})$ associated to the conflict graph $\mathfrak{G}$, is to construct a set packing relaxation of the ASP. To this purpose, consider the function $\pi : \mathbb{R}^A \to \mathbb{R}^{\mathfrak{V}}$ defined as

$$(1) \qquad \pi_{\mathfrak{v}}(x) := \sum_{ij \in \mathfrak{v}} x_{ij} - (|\mathfrak{v}| - 1) \qquad \forall \text{acyclic arc sets } \mathfrak{v} \in \mathfrak{V}.$$

$\pi(x)$ is integral for all integral $x \in \mathbb{R}^A$. Moreover, for every incidence vector $x \in P_{\mathrm{ASP}}$ of an acyclic arc set $\mathrm{supp}(x)$ in D_n, we have that $\pi(x)$ attains its maximum value of one in component $\pi_{\mathfrak{v}}(x)$ if and only if $\mathfrak{v}$ is contained in $\mathrm{supp}(x)$. Since two conflicting acyclic arc sets cannot simultaneously be contained in $\mathrm{supp}(x)$, we have that

$$\pi_{\mathfrak{u}}(x) + \pi_{\mathfrak{v}}(x) \leq 1 \iff \mathfrak{u}\mathfrak{v} \in \mathfrak{E} \qquad \forall x \in P_{\mathrm{ASP}} \cap \mathbb{Z}^A$$

and, by convexity, also for all $x \in P_{\mathrm{ASP}}$. This argument proves that $\breve{P}_{\mathrm{SSP}}(\mathfrak{G})$ is a set packing relaxation of P_{ASP} in the sense that

2.1 Lemma (Set Packing Relaxation of the ASP) $\pi(P_{\mathrm{ASP}}) \subseteq \breve{P}_{\mathrm{SSP}}(\mathfrak{G}(D_n))$.

Note that it is not possible to replace $\breve{P}_{\mathrm{SSP}}$ with P_{SSP}, because the components of π can take negative values. More precisely, $\pi(x)$ is in general *not* the incidence vector of a stable set in $\breve{P}_{\mathrm{SSP}}(\mathfrak{G})$, but $\max\{\mathbf{0}, \pi(x)\}$, with the maximum taken in every component, is.

Lemma 2.1 allows us to *expand* an inequality $a^T\pi \le \alpha$ that is valid for $\breve{P}_{\mathrm{SSP}}$ into the inequality $a^T\pi(x) \le \alpha$ for P_{ASP}. Our next Theorem states that, with this terminology, the fence and Möbius ladder inequalities are expansions of clique and odd cycle inequalities, respectively.

2.2 Theorem (Fence and Möbius Ladder Inequalities)
Let D_n be the complete digraph on n nodes, P_{ASP} the corresponding acyclic subdigraph polytope, $\mathfrak{G}$ the conflict graph associated to D_n, and $\breve{P}_{\mathrm{SSP}}(\mathfrak{G})$ the set packing relaxation of P_{ASP}.

(i) Every k-fence inequality for P_{ASP} is the expansion of a clique inequality for $\breve{P}_{\mathrm{SSP}}(\mathfrak{G})$.

(ii) Every Möbius ladder inequality for P_{ASP} is the expansion of an odd cycle inequality for $\breve{P}_{\mathrm{SSP}}(\mathfrak{G})$.

Proof.
(i) Let F_k be a k-fence. The acyclic arc sets F_k^i, $i = 1, \ldots, k$, defined on the previous page, form a clique in $\mathfrak{G}$, see the discussion on the previous page. An expansion of the corresponding clique inequality yields the desired k-fence inequality:

$$\begin{aligned}
& \sum_{i=1}^{k} \pi_{F_k^i}(x) \le 1 \\
\iff\ & \sum_{i=1}^{k}\left(\sum_{ij\in F_k^i} x_{ij} - (|F_k^i| - 1)\right) = \sum_{i=1}^{k}\left(\sum_{ij\in F_k^i} x_{ij} - (k-1)\right) = \sum_{ij\in F_k} x_{ij} - k^2 + k \le 1 \\
\iff\ & \sum_{ij\in F_k} x_{ij} \le k^2 - k + 1.
\end{aligned}$$

(ii) Let M be a Möbius ladder consisting of an odd number $2k+1$ of dicycles $C_0, \ldots, C_{2k}$ such that C_i and C_{i+1} have a dipath P_i in common. The argument on the previous page showed that the dipaths $C_i \setminus P_i$ form an odd cycle of $2k+1$ acyclic arc sets in $\mathfrak{G}$. Expanding the corresponding odd cycle inequality for $\breve{P}_{\mathrm{SSP}}(\mathfrak{G})$, one obtains the Möbius ladder inequality for M:

$$\begin{aligned}
& \sum_{i=0}^{2k} \pi_{C_i\setminus P_i}(x) \le k \\
\iff\ & \sum_{i=0}^{2k}\left(\sum_{ij\in C_i\setminus P_i} x_{ij} - (|C_i\setminus P_i| - 1)\right) \le k \\
\iff\ & \sum_{i=0}^{2k}\sum_{ij\in C_i\setminus P_i} x_{ij} \le \left(\sum_{i=0}^{2k} |C_i\setminus P_i|\right) - (k+1).
\end{aligned}$$

□

Fence and Möbius ladder inequalities have been discussed in a number of different frameworks in the literature. Euler, Jünger & Reinelt [1987] interpret fences and Möbius ladders without arc repetitions as *generalized cliques* and *generalized odd cycles* of an *independence system relaxation* of the ASP. Müller & Schulz [1995, 1996] give cutting plane proofs of fence and Möbius ladder inequalities in the more general context of *transitive packing*. Caprara & Fischetti [1996] give a derivation of Möbius ladder inequalities in terms of $\{0, \frac{1}{2}\}$ *Chvátal-Gomory cuts*. The last two constructions work for Möbius ladders with arc repetitions and yield a class that is "in the middle" between Grötschel, Jünger & Reinelt [1985a]'s Möbius ladder inequalities and our's, depending on the number of dipaths that contain a given repeated arc.

Separation of fence inequalities was shown to be $\mathcal{NP}$-hard by Müller [1993]. We remark that the orthogonality constraints of Grötschel, Lovász & Schrijver [1988] (or the $N_+\big(\mathrm{FR}(\mathfrak{G})\big)^*$ matrix cuts of Lovász & Schrijver [1991]), generated from the restriction $\mathfrak{G}\big[\{F_k^i\}\big]$ of $\mathfrak{G}$ to the acyclic arc sets F_k^i, yield a generalization of the k-fence inequalities. For bounded k, these constraints are polynomial time separable.

Looking at the *separation of Möbius ladder inequalities*, we notice that the construction that we presented to prove Theorem 2.2 (ii) yields a class of "odd cycle of dipaths inequalities" that subsumes the Möbius ladder inequalities. For this family we can devise a polynomial time separation algorithm, even though the number of dipaths is exponential and their length is arbitrary. The idea is to construct a most violated Möbius ladder out of properly interlinked longest dipaths.

Suppose that M is an odd cycle of dipaths $C_i \setminus P_i$ that induces a violated inequality and consider the dipath P_i linking the two (successive) dicycles C_i and C_{i+1}. We can isolate the contribution of P_i in the constraint as

$$\left(\sum_{j\neq i}\sum_{ij\in P_j} x_{ij}\right) - \left(|P_i| - \sum_{ij\in P_i} x_{ij}\right) > \left(\sum_{j\neq i}|P_j|\right) - (k+1)/2.$$

(Note that we have $>$ because the constraint was, by assumption, violated.) If we replace P_i by a dipath P that is shorter with respect to the length function

$$|P| - \sum_{ij\in P} x_{ij} = \sum_{ij\in P}(1 - x_{ij}), \tag{2}$$

we get a more violated cycle of dipaths inequality. If we think of any dicycle C_i as being composed out of four dipaths, namely the dipath $P_i^1 = P_i$, that C_i has in common with the succeeding dicycle, the dipath P_i^2 from P_i^1's head to the dipath $P_{i-1} = P_i^3$, that C_i has in common with the preceeding dicycle, and the remaining dipath P_i^4 from P_i^3's head to P_i^1's tail, the same argument holds for any of these dipaths. This observation allows us to show

2.3 Theorem (Polynomial Time Separability of Odd Cycle of Dipaths Inequalities)
Let D_n be the complete digraph on n nodes and P_{ASP} the associated acyclic subdigraph polytope. Suppose that $x \in \mathbb{Q}^A$ satisfies the constraints (ASP) (ii)–(iv). Then:

Odd cycle of dipaths inequalities can be separated in polynomial time.

Proof.
Using Dijkstra's algorithm, we can compute a shortest dipath $P(u, v)$ with respect to the length (2) from any node u to any node v of D_n. This yields a polynomial number of shortest dipaths and, moreover, (not every, but) *a* most violated cycle of dipaths will consist of a polynomial number of these shortest dipaths.

We can find a set of them forming an odd cycle of dipaths as follows. We think of all dipaths $P(u, v)$ as a possible common dipath P_i of two successive dicycles C_i and C_{i+1} in a cycle of dipaths. To get the dipaths $C_i \setminus P_i$ as the pieces of the cycle, we compute for any two such dipaths P_i and P_j the (2)-shortest dipath $P_i\langle P_j\rangle$ that starts at P_i's head, contains P_j, and ends at P_i's tail. Such a dipath $P_i\langle P_j\rangle$ will link (on P_j) properly with another dipath $P_j\langle P_k\rangle$ to form a cycle of dipaths. Computation of the $P_i\langle P_j\rangle$ can be performed in polynomial time and yields, in particular, a polynomial number of dipaths.

We can now construct a digraph that has these dipaths $P_i\langle P_j\rangle$ as its nodes with node weights equal to the path lengths (2), and arcs that connect $P_i\langle P_j\rangle$ to all $P_j\langle P_k\rangle$. The node weights on an arc never exceed one because x satisfies the dicycle inequalities (ASP) (ii), a most violated cycle of dipaths inequality corresponds to a most violated odd dicycle inequality in the $P_i\langle P_j\rangle$-digraph, and we can find a most violated odd dicycle inequality there with the algorithm described in Grötschel, Jünger & Reinelt [1985a]. $\square$

2.4 Corollary (Separation of Möbius Ladder Inequalities)
A superclass of the Möbius ladder inequalities can be separated in polynomial time.

To discuss the results on Möbius ladder separation of the literature, we draw the readers attention to a subtile difference between the ASP and the LOP: While the length of the dicycles in a facet defining Möbius ladder inequality (as defined in this paper) for the *acyclic subdigraph polytope* can be arbitrarily large, the constraint can only define a facet for the *linear ordering polytope* if the length of each dicycle is at most four, see Grötschel, Jünger & Reinelt [1985a]. For the LOP, one can thus restrict Corollary 2.4 to the case $|C_i| \leq 4$ and then it also follows from Müller & Schulz [1995] and Caprara & Fischetti [1996]. For the ASP, Caprara & Fischetti [1996] showed polynomial time separability of Möbius ladder inequalities where all dicycles have at most constant length.

3 The Clique Partitioning, the Multi-, and the Max Cut Problem

In this section, we investigate set packing relaxations of combinatorial optimization problems in connection with cuts: The clique partitioning, the k-multicut, and the max cut problem. We will see that the 2-chorded cycle inequalities for the clique partitioning polytope can be seen as cycles of "lower triangle" inequalities. An analogous construction for cycles of "upper triangle" inequalities is related to the circulant inequalities for the max cut polytope. As a reference to the clique partitioning problem, we suggest Grötschel & Wakabayashi [1990], for the multicut and the max cut problem Deza & Laurent [1997].

The three cut problems of this section come up on a complete graph $K_n = (V, E)$ on n nodes with integer weights $w : E \to \mathbb{Z}$ on its edges. The *clique partitioning problem* (CPP) is to find a partition of V into an arbitrary number k of cliques $V = C_1 \uplus \ldots \uplus C_k$ (where $\uplus$ denotes a union of disjoint sets), such that the sum of the weights of the edges that run between different cliques is maximal. In other words, we are trying to find a *multicut* $\delta(C_1 : \cdots : C_k)$ of maximum weight, where the number k of (non-empty) members C_i of the *clique partition* $C_1 \uplus \ldots \uplus C_k$ is arbitrary. One obtains the *k-multicut problem* (k-MCP) from this formulation by restricting the number of cliques to be less than or equal to some given number k, and the *max cut problem* (MCP) by prescribing $k = 2$. Thus, any (max) cut is a k-multicut ($k \geq 2$), and any k-multicut comes from a clique partition. We remark that the CPP is commonly stated in an equivalent version to find a clique partition that minimizes the sum of the edge weights *inside* the cliques.

Integer programming formulations of the clique partitioning and the k-multicut problem read

$$
\begin{array}{lrl}
\max & \displaystyle\sum_{ij \in E} w_{ij} x_{ij} & \\
 & & \\
\text{(ii)} & x_{ij} - x_{jk} - x_{ik} \leq 0 & \forall \{i,j,k\} \subseteq V \\
\text{(iii)} & -x_{ij} \leq 0 & \forall ij \in E \\
\text{(iv)} & x_{ij} \leq 1 & \forall ij \in E \\
\text{(v)} & x_{ij} \in \mathbb{Z} & \forall ij \in E
\end{array}
\qquad \text{(CPP)}
$$

$$
\begin{array}{lrl}
\max & \displaystyle\sum_{ij \in E} w_{ij} x_{ij} & \\
\text{(i)} & \displaystyle\sum_{ij \in E(W)} x_{ij} \leq |E(W)| - 1 & \forall W \subseteq V : |W| = k+1 \\
\text{(ii)} & x_{ij} - x_{jk} - x_{ik} \leq 0 & \forall \{i,j,k\} \subseteq V \\
\text{(iii)} & -x_{ij} \leq 0 & \forall ij \in E \\
\text{(iv)} & x_{ij} \leq 1 & \forall ij \in E \\
\text{(v)} & x_{ij} \in \mathbb{Z} & \forall ij \in E.
\end{array}
\qquad (k-\text{MCP})
$$

Setting k to 2, inequalities (k-MCP) (i) turn out to be the "upper triangle" inequalities $x_{ij} + x_{jk} + x_{ik} \leq 2$ for all $\{i,j,k\} \subseteq V$ and (2-MCP) is an integer programming formulation for the max-cut problem (we speak of upper triangle inequalities because their normal vectors are oriented "upward" such that the induced face is on the "upside" of the polytope). For $k = n$, on the other hand, (k-MCP) (i) becomes void and (n-MCP) coincides with (CPP). Hence, (CPP) is a relaxation of (k-MCP) which in turn is a relaxation of (MCP) and the associated polytopes P_{CPP}, $P_{k-\text{MCP}}$, and P_{MCP} satisfy

$$P_{\text{CPP}} \supseteq P_{k-\text{MCP}} \supseteq P_{\text{MCP}}.$$

In particular, any valid inequality for the clique partitioning polytope is also valid for the k-multicut and the max-cut polytope. One such family are the *2-chorded cycle inequalities* of Grötschel & Wakabayashi [1990].

A *2-chorded cycle* is an odd cycle C of K_n together with its set of 2-chords $\overline{C}$, see Figure 5. The associated inequality states that

$$\sum_{ij \in \overline{C}} x_{ij} - \sum_{ij \in C} x_{ij} \leq (|C| - 1)/2.$$

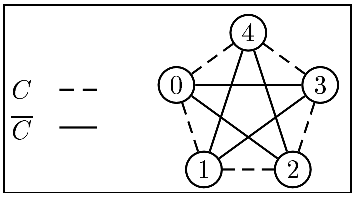


Figure 5: A 2-Chorded Cycle.

Müller [1996], and later Caprara & Fischetti [1996], proved that 2-chorded cycle inequalities can be separated in polynomial time. We will show next that this class arises from odd cycle inequalities of a set packing relaxation of the clique partitioning (or k-multicut or max cut) problem. Our arguments yield an alternative proof for the polynomial time separability of this class.

The relaxation involves a "lower triangle" *conflict graph* $\mathfrak{G}_\Delta(K_n) = (\mathfrak{V}_\Delta, \mathfrak{E}_\Delta)$. $\mathfrak{V}_\Delta$ consists of all *ordered* triples $(i,j,k) \in V^3$ of distinct nodes of K_n, the edges $\mathfrak{E}$ of $\mathfrak{G}$ are of the form $(i,j,k)(l,i,j)$, $(i,j,k)(l,j,i)$, $(i,j,k)(l,i,k)$, and $(i,j,k)(l,k,i)$ (the meaning of this definition will become clear in a second).

To construct a set packing relaxation of the clique partitioning problem with this graph, we define a mapping $\pi : \mathbb{R}^E \to \mathbb{R}^{\mathfrak{V}_\Delta}$ as

$$\pi_{(i,j,k)}(x) := x_{ij} - x_{jk} \qquad \forall \text{ordered triples } (i,j,k) \in \mathfrak{V}_\Delta.$$

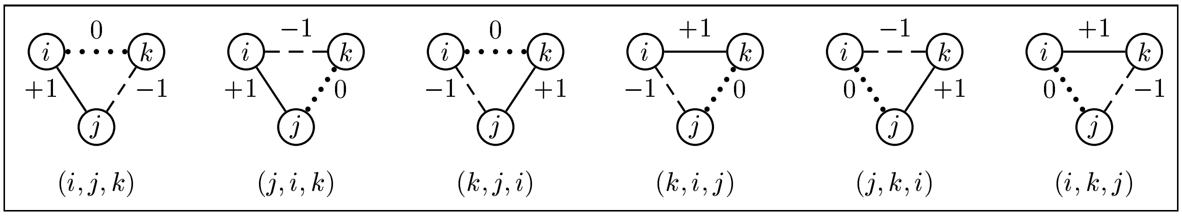


Figure 6: Labelled Triangles.

$\pi_{(i,j,k)}(x)$ is integral if $x \in \mathbb{R}^E$ is integral. Moreover, for every multicut $x \in P_{\text{CPP}}$, the component $\pi_{(i,j,k)}(x)$ attains its maximum value of one if and only if the nodes j and k belong to the same clique ($x_{jk} = 0$), but node i does not ($x_{ij} = x_{ik} = 1$). The reader may think of the triples (i, j, k) as "edge-labelled triangles" as shown in Figure 6; then, it is easy to see that

$$\pi_{\mathfrak{u}}(x) + \pi_{\mathfrak{v}}(x) \leq 1 \iff \mathfrak{uv} \in \mathfrak{E}_\Delta \qquad \forall x \in P_{\text{CPP}} \cap \mathbb{Z}^E$$

and thus for all $x \in P_{\text{CPP}}$. In other words, $\mathfrak{E}_\Delta$ was defined in such a way that two triples are joined by an edge if and only if it is impossible that both attain their maximum value of one under π simultaneously. This argument shows that $P_{\text{SSP}}(\mathfrak{G}_\Delta)$ is a "lower triangle" set packing relaxation of P_{CPP}:

3.1 Lemma (Set Packing Relaxation of the CPP) $\pi(P_{\text{CPP}}) \subseteq P_{\text{SSP}}\big(\mathfrak{G}_\Delta(K_n)\big)$.

The construction is called a "lower triangle set packing relaxation", because one obtains the components $\pi_{(i,j,k)}(x) = x_{ij} - x_{jk} \leq 1$ of π from the lower triangle inequalities (CPP) (ii) by setting $x_{ik} = 1$:

$$x_{ij} - x_{jk} - x_{ik} \leq 0 \iff x_{ij} - x_{jk} \leq x_{ik}.$$

We are now ready to state our result that the 2-chorded cycle inequalities are expansions (see the definition on page 5) of odd cycle inequalities of $\check{P}_{\text{SSP}}(\mathfrak{G}_\Delta)$.

3.2 Theorem (2-Chorded Cycle Inequalities)
Let K_n be the complete graph on n nodes, P_{CPP} the corresponding clique partitioning polytope, $\mathfrak{G}_\Delta$ the lower triangle conflict graph, and $\check{P}_{\text{SSP}}(\mathfrak{G}_\Delta)$ the lower triangle set packing relaxation of P_{CPP}.

Every 2-chorded cycle inequality for P_{CPP} is the expansion of an odd cycle inequality for $\check{P}_{\text{SSP}}(\mathfrak{G}_\Delta)$.

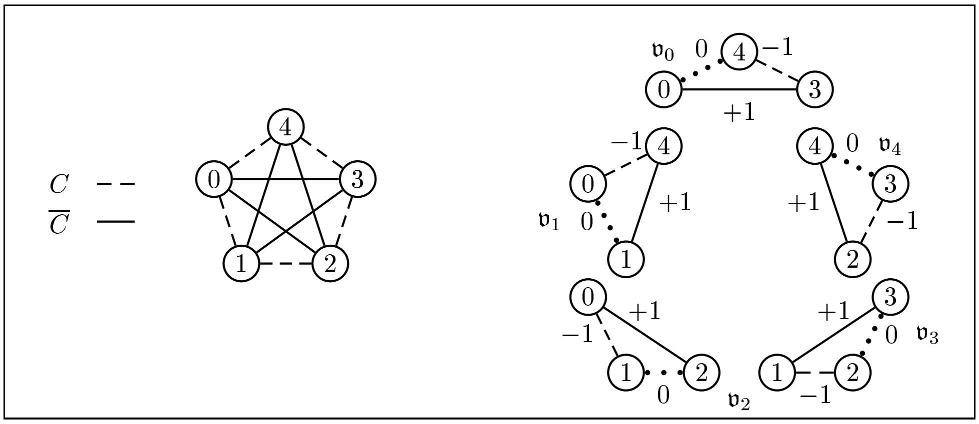


Figure 7: An Odd-Cycle of Lower Triangle Inequalities.

Proof.
Let $C \cup \overline{C}$ be a 2-chorded cycle in K_n with node set $\{0, \ldots, 2k\}$. By definition, $C = \{ij : i = 0, \ldots, 2k,\ j = i + 1\}$ and $\overline{C} = \{ij : i = 0, \ldots, 2k,\ j = i + 2\}$ (where indices are taken modulo $2k + 1$).

Consider the $2k + 1$ triples $\mathfrak{v}_i := (i, i - 2, i - 1)$, $i = 0, \ldots, 2k$ (indices modulo $2k + 1$). One verifies that $\mathfrak{v}_i\mathfrak{v}_{i+1} \in \mathfrak{E}$ are in conflict and form the edge set of an odd cycle in $\mathfrak{G}_\Delta$, see Figure 7 for an example. The associated odd cycle inequality expands to the 2-chorded cycle inequality in question:

$$\sum_{i=0}^{2k} \pi_{(i,i-2,i-1)}(x) = \sum_{i=0}^{2k} x_{(i,i-2)} - x_{(i-2,i-1)} = \sum_{ij \in \overline{C}} x_{ij} - \sum_{ij \in C} x_{ij} \leq (|C| - 1)/2.$$

□

Calling the expansions of odd cycle inequalities for $\check{P}_{\text{SSP}}(\mathfrak{G}_\Delta)$ *inequalities from odd cycles of lower triangle inequalities* and noting $|\mathfrak{V}_\Delta| = O(n^3)$, we obtain

3.3 Corollary (Separation of Inequalities from Odd Cycles of Lower Triangle Inequalities) *Let K_n be the complete graph on n nodes and P_{CPP} the associated clique partitioning polytope. Suppose $x \in \mathbb{Q}^E$ satisfies the constraints (CPP) (ii)–(v). Then:*

Odd cycle of lower triangle inequalities can be separated in polynomial time.

3.4 Corollary (Separation of 2-Chorded Cycle Inequalities) *A superclass of the 2-chorded cycle inequalities can be separated in polynomial time.*

Note that the conflicts between two successive triples $\mathfrak{v}_i = (i, i-2, i-1)$ and $\mathfrak{v}_{i+1} = (i+1, i-1, i)$ in a 2-chorded cycle stem from the common edge connecting nodes i and $i-1$, that has a coefficient of -1 in $\pi_{\mathfrak{v}_{i+1}}$ and 0 in $\pi_{\mathfrak{v}_i}$. But conflicts arise also from common edges with $+1$ and -1 coefficients. Thus, besides possible node/edge repetitions and the like, odd cycle of lower triangle inequalities give rise to inequalities that do not correspond to 2-chorded cycle inequalities.

So far we have studied inequalities from "pairwise conflicts of lower triangle inequalities". In the case of the max cut problem, the constraints (2-MCP) (i) form a class of "upper triangle inequalities"

$$x_{ij} + x_{jk} + x_{ik} \leq 2 \qquad \forall i, j, k \in V.$$

Analogous to the lower triangle case, we will now construct "inequalities from odd cycles of upper triangle inequalities" for the max cut polytope. These constraints are related to the $C(2k+1, 2)$-*circulant inequalities* of Poljak & Turzik [1992].

A $C(2k+1, 2)$-*circulant* is identical to a 2-chorded cycle on an odd number of $2k+1$ nodes. We distinguish circulants $C(2k+1, 2)$ with *odd* k and with *even* k. The associated inequalities are

$$\sum_{ij \in C(2k+1,2)} x_{ij} \leq 3k+1, \qquad \text{if } k \bmod 2 = 1 \qquad\qquad \sum_{ij \in C(2k+1,2)} x_{ij} \leq 3k, \qquad \text{if } k \bmod 2 = 0.$$

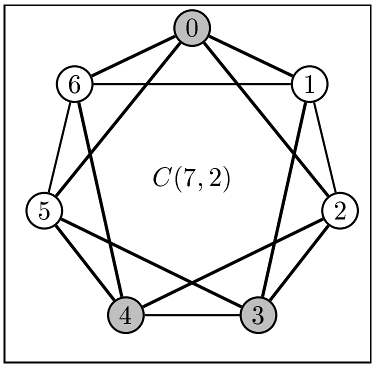


Figure 8: The Odd-k Circulant $C(7, 2)$.

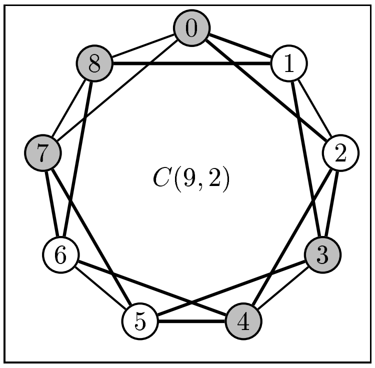


Figure 9: The Even-k Circulant $C(9, 2)$.

Even-k circulant inequalities have been introduced by Poljak & Turzik [1992]. In this case, the right-hand side is $3k$, whereas the odd case requires an increase of one in the right-hand side: Figure 8 shows the odd-k circulant $C(7, 2)$ ($k = 3$); the white and gray nodes form the shores of a cut with $10 = 3{\cdot}3+1 > 3{\cdot}3 = 9$ edges highlighted. Alternatingly putting pairs of successive nodes on the left and on the right shore of the cut except for the first node, one verifies that the right-hand side $3k+1$ is best possible for odd k. Figure 9 shows a tight configuration for the case where k is even. (A rigorous proof for the validity of these constraints will follow from the upcoming discussion.)

We will show now that the circulant inequalities can be seen as "strengthened" odd cycle inequalities of an appropriate "upper triangle" set packing relaxation of the max cut problem; strengthened means that for even k, the right-hand side of the cycle of upper triangles inequality exceeds the right-hand side of the corresponding circulant inequality by one. Our considerations allow to design a polynomial time algorithm for separating cycles of upper triangle inequalities. Poljak & Turzik [1992], on the other hand, have shown that separation of the *exact* class of $C(2k+1, 2)$-circulant inequalities is $\mathcal{NP}$-hard.

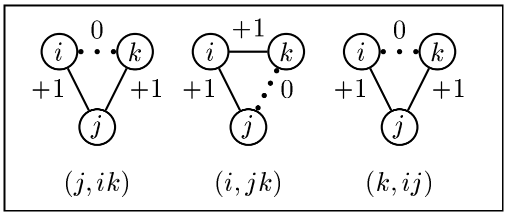


Figure 10: Labeling upper triangles.

As usual, the upper triangle set packing relaxation is based on an upper triangle conflict graph $\mathfrak{G}^\Delta(K_n) = (V^\Delta, E^\Delta)$. This time, $\mathfrak{V}^\Delta$ consists of all 2-tuples $(i, jk) \in V \times E$ such that $i \notin jk$, while $\mathfrak{E}^\Delta$ is the set of all $(i, jk)(j, kl)$. To construct a set packing relaxation of P_{MCP} by means of this graph, we introduce a function $\pi : \mathbb{R}^E \to \mathbb{R}^{\mathfrak{V}^\Delta}$ defined as

$$\pi_{(i,jk)}(x) := x_{ij} + x_{ik} - 1.$$

One gets $\pi_{(i,jk)}(x) \leq 1$ from the upper triangle inequality

$$x_{ij} + x_{ik} + x_{jk} \leq 2 \iff x_{ij} + x_{ik} - 1 \leq 1 - x_{jk}$$

by setting $x_{jk} = 0$, hence the name "upper triangle" conflict graph. This also proves that $\pi_{(i,jk)}(x)$ attains its maximum value of one if and only if node i is on one side of the cut, while nodes j and k are on the other. Again, one may think of the nodes i, j, and k forming a triangle with the edges labelled as indicated in Figure 10 and sees that

$$\pi_{\mathfrak{u}}(x) + \pi_{\mathfrak{v}}(x) \leq 1 \iff \mathfrak{u}\mathfrak{v} \in \mathfrak{E}^\Delta.$$

Thus, $\breve{P}_{\text{SSP}}(\mathfrak{G}^\Delta)$ is an "upper triangle" set packing relaxation of P_{MCP}.

3.5 Lemma (Set Packing Relaxation of the MCP) $\pi(P_{\text{MCP}}) \subseteq \breve{P}_{\text{SSP}}\big(\mathfrak{G}^\Delta(K_n)\big)$.

This construction yields the circulant inequalities for $k \bmod 2 = 1$ as expansions of odd cycle inequalities for the upper triangle set packing relaxation of the max cut polytope. The case $k \bmod 2 = 0$ can be settled by *strengthening* the associated odd cycle inequality, i.e., one can decrease the right-hand side of the inequality by one and the inequality remains still valid.

3.6 Theorem (Circulant Inequalities)
Let K_n be the complete graph on n nodes, P_{MCP} the corresponding max cut polytope, $\mathfrak{G}^\Delta$ the upper triangle conflict graph, and $\breve{P}_{\text{SSP}}(\mathfrak{G}^\Delta)$ the upper triangle set packing relaxation of P_{MCP}.

(i) Every odd-k circulant inequality for P_{MCP} is the expansion of an odd cycle inequality for $\breve{P}_{\text{SSP}}(\mathfrak{G}^\Delta)$.

(ii) Every even-k circulant inequality for P_{MCP} is the expansion of a strengthened odd cycle inequality for $\breve{P}_{\text{SSP}}(\mathfrak{G}^\Delta)$.

Proof.
(i) Let $C(2k+1, 2)$ be an odd-k circulant with node set $\{0, \ldots, 2k+1\}$. Consider the $2k+1$ 2-tuples $\mathfrak{v}_i := \big(i, (i+1, i+2)\big)$ (with indices taken modulo $2k+1$). One verifies that the tuples $\mathfrak{v}_i$ and $\mathfrak{v}_{i+1}$ are in conflict, i.e., $\mathfrak{v}_i\mathfrak{v}_{i+1} \in \mathfrak{E}^\Delta$, and form an odd cycle in $\mathfrak{G}^\Delta$, see Figure 11. The associated odd cycle inequality expands to a circulant inequality:

$$\begin{aligned} & \sum_{i=0}^{2k} \pi_{\big(i,(i+1,i+2)\big)}(x) \leq k \\ \iff & \sum_{i=0}^{2k} (x_{(i,i+1)} + x_{(i,i+2)} - 1) \leq k \\ \iff & \sum_{ij \in C(2k+1,2)} x_{ij} \leq 3k+1. \end{aligned}$$

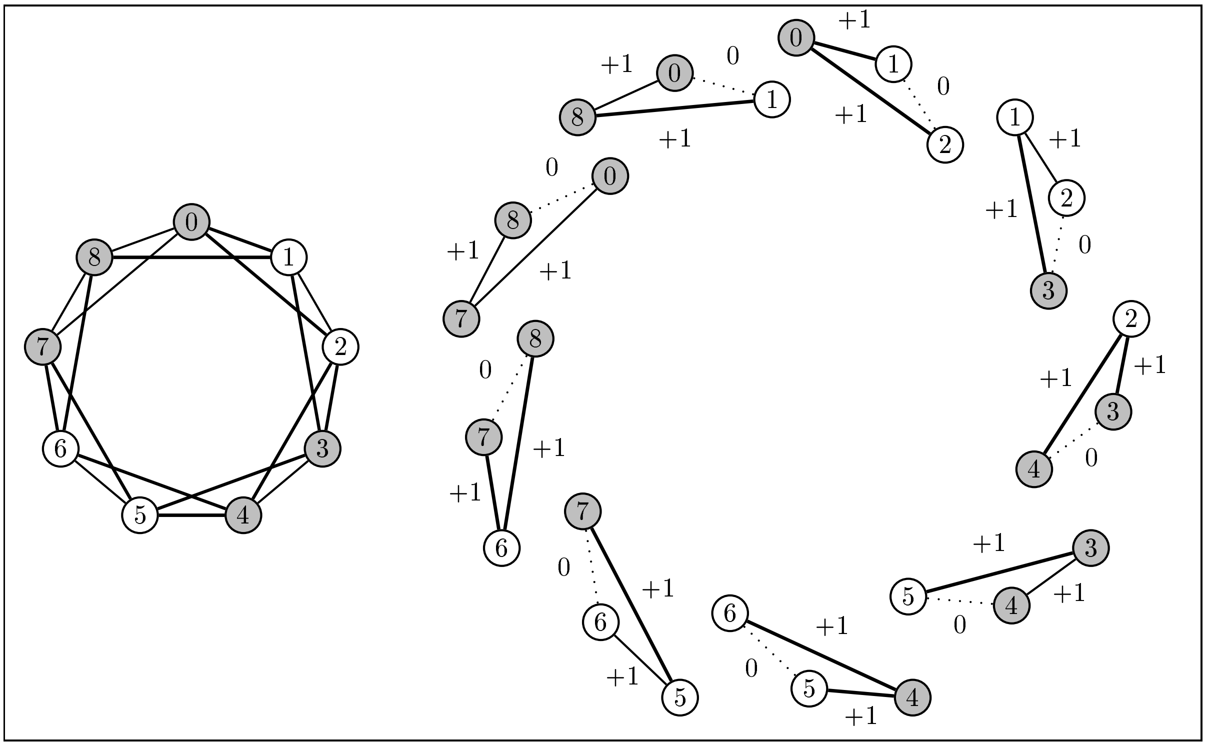


Figure 11: An Odd Cycle of Upper Triangles Inequalities.

(ii) The even case is analogous to the odd case. To see that one can reduce the right-hand side of the odd cycle inequality by one and $\sum_{i=0}^{2k} \pi_{(i,(i+1,i+2))}(x) \leq k-1$ is still valid, suppose this is not so and let $x \in P_{\mathrm{MCP}}$ be the incidence vector of cut that violates this constraint. Now, $\max\{\mathbf{0}, \pi(x)\}$ is the incidence vector of a stable set in $\mathfrak{G}^{\Delta}$, and, clearly, this vector must be tight for the (unstrengthened) odd cycle inequality. This means that we have k 2-tuples with $\max\{\mathbf{0}, \pi_{(i,(i+1,i+2))}(x)\} = 1$, and $k+1$ 2-tuples with $\max\{\mathbf{0}, \pi_{(i,(i+1,i+2))}(x)\} = 0$, that are arranged in such a way that the two types appear alternatingly except for one time, where we have two "zeros" next to each other. Looking at $\max\{\mathbf{0}, \pi_{(i,(i+1,i+2))}(x)\} = 1$, we see that node i must be on one side of the cut while nodes $i+1$ and $i+2$ must be on the other. The next "one" $\max\{\mathbf{0}, \pi_{(i+2,(i+3,i+4))}(x)\} = 1$ forces nodes $i+3$ and $i+4$ to be on the same side as i. Starting without loss of generality at $\pi_{(0,(1,2))} = 1$ and continuing k times like this, all nodes of the circulant are assigned to one side of the cut or another in a unique way. When k is even, this results in nodes $2k-1$, $2k$, and 0 ending up on the same side such that $\pi_{(2k-1,(2k,0))}(x)\} = -1$, see the right side of Figure 11 for an example; but then $\sum_{i=0}^{2k} \pi_{(i,(i+1,i+2))}(x) \leq k-1$, a contradiction. $\square$

Calling the expansion of an odd cycle inequality for $\check{P}_{\mathrm{SSP}}(\mathfrak{G}^{\Delta})$ an *inequality from an odd cycle of upper triangle inequalities*, we obtain

3.7 Corollary (Separation of Inequalities from Cycles of Upper Triangle Inequalities)
Let K_n be the complete graph on n nodes and P_{MCP} the associated max cut polytope. Suppose $x \in \mathbb{Q}^E$ satisfies the constraints (2-MCP) (i)–(v). Then:

Odd cycle of upper triangle inequalities can be separated in polynomial time.

3.8 Corollary (Separation of Circulant Inequalities)

(i) A superclass of odd-k $C(2k+1,2)$ circulant inequalities can be separated in polynomial time.

(ii) A superclass of even-k $C(2k+1,2)$ circulant inequalities with their right-hand sides increased by one can be separated in polynomial time.

We remark again that Poljak & Turzik [1992] have shown that it is $\mathcal{NP}$-complete to determine whether a graph contains a $C(2k+1,2)$ circulant and thus, separation of the *exact* class of $C(2k+1,2)$ circulant inequalities is $\mathcal{NP}$-hard.

4 The Set Packing Problem

We have demonstrated in the examples of the preceeding sections that certain combinatorial optimization problems have interesting set packing relaxations. Perhaps a bit surprising, we show now that the set packing problem *itself* also has interesting set packing relaxations! These considerations yield alternative derivation and separation techniques for several classes of wheel inequalities, including two classes introduced by Barahona & Mahjoub [1994] and Cheng & Cunningham [1997], as well as a new class of cycle of cycles inequalities. A survey on results for the set packing problem can be found in Grötschel, Lovász & Schrijver [1988].

The examples of this sections are based on a "rank" set packing relaxation that we introduce now. Given a set packing problem (SPP) on a graph $G = (V, E)$, the associated conflict graph $\mathfrak{G} = (\mathfrak{V}, \mathfrak{E})$ of the relaxation has the set $\mathfrak{V} := \{G[H] : H \subseteq V\}$ of all node induced subgraphs of G as its nodes. In order to define the set of edges, we consider the mapping $\pi : \mathbb{R}^V \to \mathbb{R}^{\mathfrak{V}}$ defined as

$$\pi_{G[H]}(x) = \sum_{i \in G[H]} x_i - (\alpha(G[H]) - 1) \qquad \forall \text{node induced subgraphs } G[H] \in \mathfrak{V},$$

where $\alpha(G[H])$ denotes the *rank*, i.e., the maximum cardinality of a stable set, of $G[H]$. We draw an edge between two subgraphs $G[H]$ and $G[W]$ if there is no stable set in G such that its restrictions to $G[H]$ and $G[W]$ are simultaneously stable sets of maximum cardinality in $G[H]$ and $G[W]$, i.e.,

$$G[H]G[W] \in \mathfrak{E} \iff \pi_{G[H]}(x) + \pi_{G[W]}(x) \leq 1 \qquad \forall x \in P_{\mathrm{SSP}}(G) \cap \mathbb{Z}^V.$$

Well known arguments show that $\breve{P}_{\mathrm{SSP}}(\mathfrak{G})$ is a set packing relaxation of P_{SSP} in the exponential space $\mathbb{R}^{\mathfrak{V}}$:

4.1 Lemma (Rank Set Packing Relaxation of the SSP) $\pi(P_{\mathrm{SSP}}) \subseteq \breve{P}_{\mathrm{SSP}}(\mathfrak{G})$.

4.1 Wheel Inequalities

One method to derive polynomial time separable expansions of inequalities from the rank relaxation is to consider subgraphs of $\mathfrak{G}$ of polynomial size. A natural idea is to restrict the set of $\mathfrak{G}$'s nodes to

$$\mathfrak{V}_k := \{G[H] : H \subseteq V : |H| \leq k\},$$

the node induced subgraphs $G[H]$ of G with bounded numbers of nodes $|H| \leq k$ for some arbitrary, but fixed bound k. The smallest interesting case is $k = 2$, where $G[H]$ ($|H \leq 2$) is either empty, a singleton, an edge, or a coedge (complement of an edge). The odd cycle inequalities that one obtains from this restricted relaxation $\breve{P}_{\mathrm{SSP}}(\mathfrak{G}[\mathfrak{V}_k])$ contain, among other classes, the *odd wheel inequalities* of the set packing polytope.

A $2k+1$-*wheel* is an odd cycle C of $2k+1$ nodes $\{0\ldots, 2k\}$, say, plus an additional node $2k+1$ that is connected to all nodes of the cycle C. C is the *rim* of the wheel, node $2k+1$ is the *hub*, and the edges connecting the node $2k+1$ and i, $i = 0, \ldots, 2k$, are called *spokes*. For such a configuration, we have that

$$kx_{2k+1} + \sum_{i=0}^{2k} x_i \leq k.$$

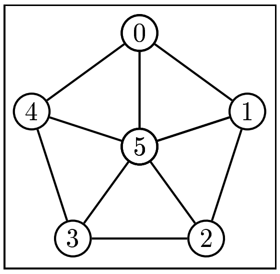


Figure 12: A 5-Wheel.

An odd wheel inequality can be obtained by a sequential lifting of the hub into the odd cycle inequality that corresponds to the rim. Trying all possible hubs, this yields a polynomial time separation algorithm for wheel inequalities. An alternative derivation is

4.2 Theorem (Odd Wheel Inequalities)
Let $G = (V, E)$ be a graph, P_{SSP} the corresponding set packing polytope, $\mathfrak{G}$ the rank conflict graph, and $\check{P}_{\mathrm{SSP}}(\mathfrak{G})$ the rank set packing relaxation of P_{SSP}.

Every odd wheel inequality for P_{SSP} is the expansion of an odd cycle inequality for $\check{P}_{\mathrm{SSP}}(\mathfrak{G}[\mathfrak{V}_2])$.

4.3 Corollary (Separation of Inequalities from Odd Cycles of Nodes, Edges, and Coedges)
Inequalities from odd cycles of nodes, edges, and coedges can be separated in polynomial time.

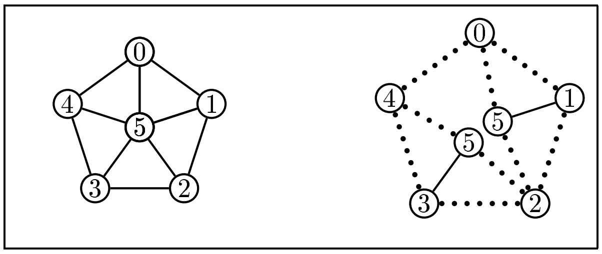


Figure 13: A Cycle of Nodes and Edges.

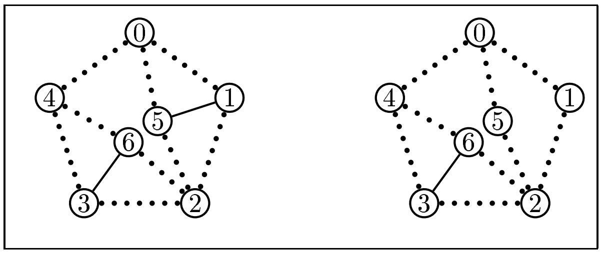
Figure 13 gives the idea for a proof of Theorem 4.2 that we omit. Instead, we show two examples of cycles of nodes, edges, and coedges, that give rise to inequalities that do not correspond to odd wheel inequalities. The cycle on the left side of Figure 14 consists of the nodes 0, 2, and 4, plus the two edges $(5, 1)$ and $(6, 3)$. It results in the inequality

$$x_0 + (x_5 + x_1) + x_2 + (x_6 + x_3) + x_4 \le 2 \iff \sum_{i=0}^{6} x_i \le 2.$$

The cycle on the right-hand side consists of the nodes 0, 2, and 4, coedge $(5, 1)$, and the edge $(6, 3)$. It gives rise to the inequality

$$x_0 + (x_5 + x_1 - 1) + x_2 + (x_6 + x_3) + x_4 \le 2 \iff \sum_{i=0}^{6} x_i \le 3.$$

$$\sum_{i=0}^{6} x_i \le 2$$



$$\sum_{i=0}^{6} x_i \le 3$$

Figure 14: Two Generalizations of Odd Wheel Inequalities.

Another generalization of odd wheel inequalities was given by Barahona & Mahjoub [1994] and Cheng & Cunningham [1997]. They introduce two classes of inequalities that have *subdivisions* of odd wheels as support graphs, where each face cycle must be odd, see Figure 15. Following Cheng & Cunningham [1997]'s terminology and denoting the endnodes of the *even spokes* (with an even number of *edges*) of an odd wheel W of this kind with some number $2k+1$ of faces by $\mathcal{E}$, the endnodes of the *odd spokes* (with an odd number of edges) by $\mathcal{O}$, and the hub by h, a *wheel inequality of type I* states that

$$(3) \qquad kx_h + \sum_{i \in W-h} x_i + \sum_{i \in \mathcal{E}} x_i \le \frac{|W| + |\mathcal{E}|}{2} - 1.$$

A second variant of *wheel inequalities* (*of type II,* associated to the same wheel) states that

$$(4) \qquad (k+1)x_h + \sum_{i \in W-h} x_i + \sum_{i \in \mathcal{O}} x_i \le \frac{|W| + |\mathcal{O}| - 1}{2}.$$

We remark that these wheels do in general not arise from cycles of subgraphs of bounded size because they contain potentially very long paths.

4.4 Theorem (Odd Wheel Inequalities)
Let $G = (V, E)$ be a graph, P_{SSP} the corresponding set packing polytope, $\mathfrak{G}$ the rank conflict graph, and $\check{P}_{\mathrm{SSP}}(\mathfrak{G})$ the rank set packing relaxation of P_{SSP}.

Every odd wheel inequality of type I and II for P_{SSP} is the expansion of an odd cycle inequality for $\check{P}_{\mathrm{SSP}}(\mathfrak{G})$.

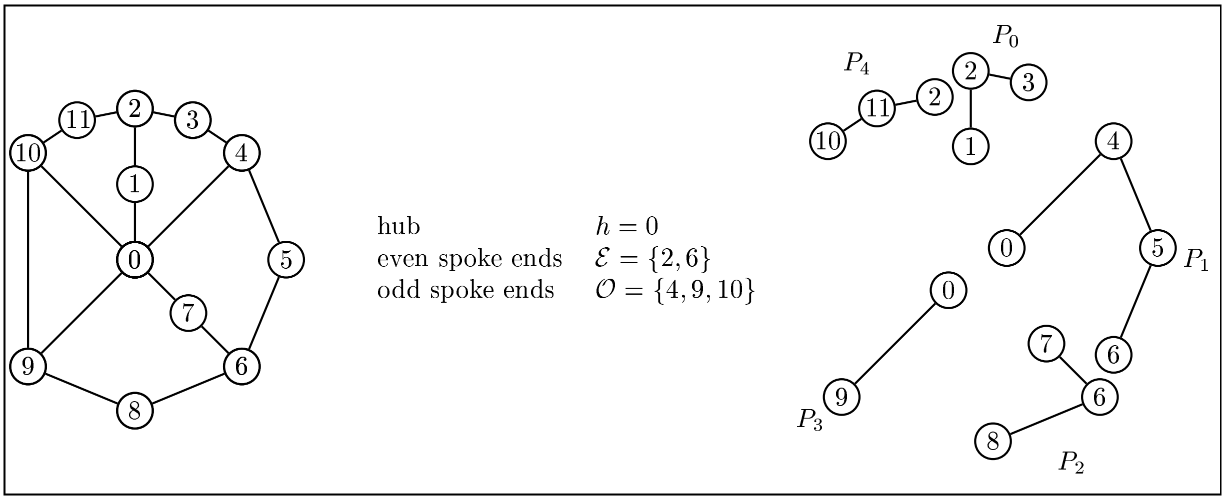


Figure 15: A 5-Wheel and a 5-Cycle of Paths of Type I.

Proof.

(i) Wheel inequalities of type I.

The idea of the proof is to obtain the wheel-inequality (3) of type I as a cycle of paths. Orienting a $2k+1$-wheel clockwise, it consists of $2k+1$ spoke paths S_i, $i = 0, \ldots, 2k$, and the same number of rim paths R_i such that R_i connects the ends of spokes S_i and S_{i+1} (indices in the proof are taken modulo $2k+1$). We can then compose the wheel from the paths

$$P_i := S_i \cup \begin{Bmatrix} R_i, & \text{if } S_{i+1} \text{ is even} \\ R_i \setminus S_{i+1}, & \text{if } S_{i+1} \text{ is odd} \end{Bmatrix} \setminus \begin{cases} \emptyset, & \text{if } i \text{ is odd} \\ \{h\}, & \text{if } i \text{ is even}, \end{cases} \qquad i = 0, \ldots, 2k,$$

see Figure 15. By definition, a path P_i consists of the spoke S_i (plus minus the hub depending on i) and the *full* rim path R_i if the end node of the next spoke (in clockwise order) is even, or the rim path R_i *without* the end of the next spoke in case this spoke is odd. In this way, the even spokes, having a coefficient of two in the wheel inequality, appear in two paths, the odd spokes in one. Finally, the hub is removed from all paths with even index. It is not hard to see that any two successive paths P_i and P_{i+1} are in pairwise conflict: The subpaths $P_i \setminus \{h\}$ with the hub removed are all odd and in pairwise conflict, and, likewise, the hub is in conflict with any of these subpaths. The odd cycle inequality corresponding to the paths P_i expands into the odd wheel inequality (3):

$$\begin{aligned}
& \sum_{i=0}^{2k} \pi_{P_i}(x) \leq k \\
\iff & \sum_{i=0}^{k} \left(\sum_{j \in P_{2i}} x_j - (|P_{2i}| - 1)/2 \right) + \sum_{i=0}^{k-1} \left(\sum_{j \in P_{2i+1}} x_j - (|P_{2i+1}| - 2)/2 \right) \leq k \\
\iff & k x_h + \sum_{j \in W \setminus \{h\}} x_j + \sum_{j \in \mathcal{E}} x_j - \frac{|W| - 1 + k + |\mathcal{E}| - (k+1) - 2k}{2} \leq k \\
\iff & k x_h + \sum_{j \in W \setminus \{h\}} x_j + \sum_{j \in \mathcal{E}} x_j \leq \frac{|W| + |\mathcal{E}| - 2k - 2}{2} + k = \frac{|W| + |\mathcal{E}|}{2} - 1.
\end{aligned}$$

(here, $|P_i|$ denotes the number of nodes in path P_i).

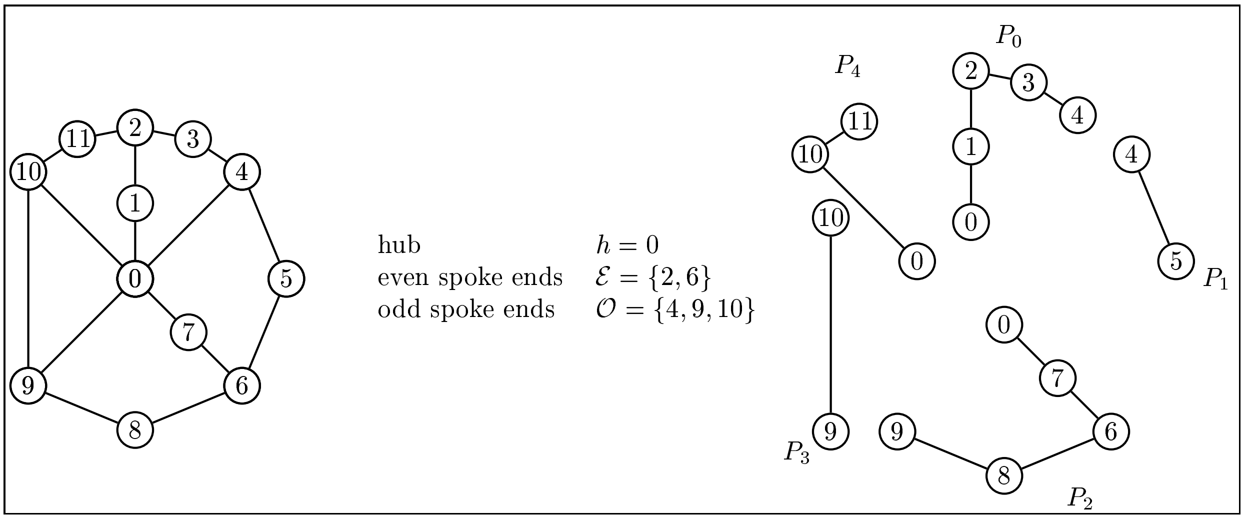


Figure 16: A 5-Wheel and a 5-Cycle of Paths of Type II.

(ii) Wheel inequalities of type II.

The wheel inequalities (4) of type II can be derived in much the same way as their relatives of type I. For the sake of completeness, we record the path decomposition

$$P_i := S_i \cup \begin{Bmatrix} R_i, & \text{if } S_{i+1} \text{ is odd} \\ R_i \setminus S_{i+1}, & \text{if } S_{i+1} \text{ is even} \end{Bmatrix} \setminus \begin{cases} \emptyset, & \text{if } i \text{ is even} \\ \{h\}, & \text{if } i \text{ is odd,} \end{cases} \qquad i = 0, \ldots, 2k.$$

One can verify that, again, any two successive paths are in conflict. A final calculation to expand the resulting odd cycle inequality yields the wheel inequality (4) of type II:

$$\begin{aligned} & \sum_{i=0}^{2k} \pi_{P_i}(x) \le k \\ \iff \quad & \sum_{i=0}^{k} \left(\sum_{j \in P_{2i}} x_j - (|P_{2i}| - 1)/2 \right) + \sum_{i=0}^{k-1} \left(\sum_{j \in P_{2i+1}} x_j - (|P_{2i+1}| - 2)/2 \right) \le k \\ \iff \quad & (k+1)x_h + \sum_{j \in W \setminus \{h\}} x_j + \sum_{j \in \mathcal{O}} x_j - \frac{|W| - 1 + (k+1) + |\mathcal{O}| - (k+1) - 2k}{2} \le k \\ \iff \quad & (k+1)x_h + \sum_{j \in W \setminus \{h\}} x_j + \sum_{j \in \mathcal{O}} x_j \le \frac{|W| + |\mathcal{O}| - 2k - 1}{2} + k = \frac{|W| + |\mathcal{O}| - 1}{2}. \end{aligned}$$

□

One can also derive polynomial time separation algorithms of much the same flavour as for the Möbius ladder inequalities; the complete procedures are given in Cheng & Cunningham [1997].

4.2 A New Family of Facets for the Stable Set Polytope

The rank relaxation of the set packing problem offers ample possibilities to define new classes of polynomially separable inequalities for the set packing problem. We discuss, as one such example, a *cycle of cycles* inequalities.

The way to construct a cycle of cycles inequality is to link an odd number $2k+1$ of odd cycles $C_0, \ldots, C_{2k}$ to a circular structure, such that any two successive cycles are in pairwise conflict, i.e., $\pi_{C_i}(x) + \pi_{C_{i+1}}(x) \le 1$ (indices taken modulo $2k+1$).

One way to do this is to select from each cycle C_i three successive nodes $L_i \subseteq C_i$ that will serve as a part of the inter-cycle links yet to be formed. The *link* L_i has the property that $\pi_{C_i}(x) = 1$ implies that at least one of the nodes in L_i is contained in the stable set $\operatorname{supp}(x)$, i.e.,

$$\pi_{C_i}(x) = 1 \Longrightarrow \sum_{j \in L_i} x_j \geq 1.$$

If we make sure that any two successive links L_i and L_{i+1} are joined by the edge set of the complete bipartite graph $K_{3,3}$, then the inequality

$$\sum_{j \in L_i} x_j + \sum_{j \in L_{i+1}} x_j \leq 1$$

holds for all incidence vectors x of stable sets in G. But then, the corresponding two successive cycles C_i and C_{i+1} are in conflict, i.e., $\pi^{C_i}(x) + \pi^{C_{i+1}}(x) \leq 1$, and the cycles C_i form an odd cycle in $\mathfrak{G}$, see Figure 17.

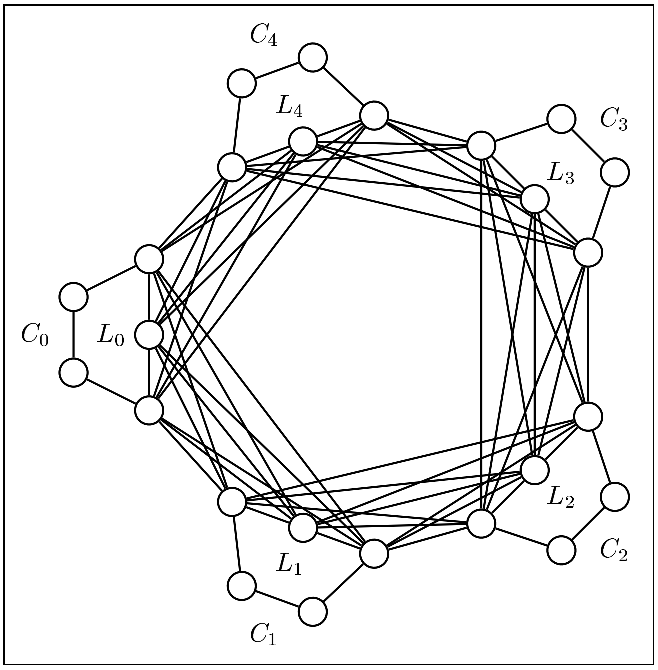


Figure 17: A 5-Cycle of 5-Cycles.

4.5 Theorem (Cycle of Cycles Inequality)
Let $G = (V, E)$ be a graph and P_{SSP} be the corresponding set packing polytope. Let C_i, $i = 0, \ldots, 2k$, be an odd cycle and $L_i \subseteq C_i$, $i = 0, \ldots, 2k$, a set of three successive nodes in C_i. Assume further that L_i and L_{i+1}, $i = 0, \ldots, 2k$, are joined by a complete $K_{3,3}$.

Then the cycle of cycles inequality

$$\sum_{i=0}^{2k} \sum_{j \in C_i} x_j \leq \sum_{i=0}^{2k} (|C_i| - 1)/2 - (k+1)$$

is valid for P_{SSP}.

Proof.

$$\begin{aligned}
& \sum_{i=0}^{2k} \pi_{C_i}(x) \leq k \\
\Longleftrightarrow\quad & \sum_{i=0}^{2k} \left(\sum_{j \in C_i} x_j - \big((|C_i| - 1)/2 - 1\big) \right) \leq k \\
\Longleftrightarrow\quad & \sum_{j \in \cup C_i} x_j \leq \sum_{i=0}^{2k} \big((|C_i| - 1)/2 - 1\big) + k = \sum_{i=0}^{2k} (|C_i| - 1)/2 - (k+1).
\end{aligned}$$

□

A cycle of cycles inequality will in general not be facet inducing, for example, if one of the cycles has a chord that does not join two nodes of its link. But one can come up with conditions that ensure this property. The most simple case is where the cycles C_i are holes, all node disjoint, and the only edges that run between different holes belong to the links, i.e., we have a “hole of holes”. In this case, the cycle of cycles inequality is easily shown to be facet inducing using standard techniques, like noting that every edge in a hole of holes is critical.

4.6 Proposition (Facet Inducing Cycle of Cyles Inequalities)
If every cycle in a cycle of cycles inequality is a hole, and the only edges that run between different holes emerge from the links, then the cycle of cycles inequalities is facet inducing.

4.7 Theorem (Separation of Cycle of Cycles Inequalities)
Cycle of cycles inequalities can be separated in polynomial time.

Proof. The number of potential links L_i is polynomial of order $O(|V|^3)$. We set up a *link graph*, that has the links as its nodes; this device will, in a second, turn out to be a subgraph of $\mathfrak{G}$. Two links are connected by an edge if and only if they are joined by a $K_{3,3}$. To assign weights to the links, we calculate for each link L_i the shortest even path P_i in G that connects the two endpoints of the link; here, shortest means shortest with respect to the length function

$$\frac{1 - x_i - x_j}{2} \qquad \forall \text{edges } ij \in E.$$

$L_i \cup P_i$ forms an odd-cycle C_i through L_i. We set the weight of link L_i to the value $\pi_{C_i}(x)$, obtain the link graph as a subgraph of $\mathfrak{G}[\{C_i\}]$ (some edges that correspond to “non-link conflicts” are possibly missing), and detect a violated odd cycle inequality in the link graph if and only if a violated cycle of cycles inequality in G exists (eventually, we first have to separate the edge inequalities of the link graph). □

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