Analysis of Algorithms

January, 1996

Note: Answer all questions. Each problem carries equal points. You must write your Social Security number on each answer book. **Do not** write your name.

1. Is it possible to use the "binary search" technique to devise an algorithm for the problem of finding $\lceil \sqrt{N} \rceil$ of a positive integer N? If yes, explain why your algorithm is binary and analyze its performance in detail. (Do not write the code—express the algorithm in words.)

2. (a) Prove that it is not possible to find the best (largest of an array of n elements) in less than $n-1$ comparisons.

   (b) What is wrong with the following argument?

   > We have shown that $n-1$ comparisons are necessary to find the best. By symmetry, $n-1$ comparisons are necessary to find the worst. Therefore $2n-2$ comparisons are necessary to find the best and the worst.

3. We know that one can build $T_n = \binom{2n}{n}/(n+1)$ binary search trees with n nodes. [T_n is also known as the Catalan number and randomly built binary trees, with equal probability T_n^{-1} are called the Catalan trees.] Show that

   (a) Probability that in a randomly built Catalan tree with n nodes the left subtree has k nodes is given by
   $$\frac{T_k T_{n-k-1}}{T_n}.$$

   (b) Let Q_n denote the average internal path length of the Catalan trees with n nodes. Show that
   $$\begin{aligned} Q_0 &= 0 \\ Q_n &= (n-1) + \sum_{1 \le k \le n} \frac{T_{k-1} T_{n-k}}{T_n} (Q_{k-1} + Q_{n-k}) \text{ for } n > 0 \end{aligned}$$

4. Let $x[1..n]$ be an array of integers in an arbitrary order, and let $a[1..n]$ be another array of distinct integers from 1 to n (namely $a[1], \ldots, a[n]$ is a *permutation* of 1, 2, $\ldots, n$). Design an $O(n \log n)$ algorithm to order the first sequence according to the order imposed by the permutation. In other words, for each i, $x[i]$ should appear in the position given in $a[i]$. For example, if $x = 17, 5, 1, 9$, and $a = 3, 2, 4, 1$ then the outcome should be $x = 9, 5, 17, 1$.

5. (a) Show that the single-source shortest path constructed by the Dijkstra's algorithm on a connected undirected graph forms a spanning tree.

   (b) Is the spanning tree form by Dijkstra's algorithm on a connected undirected graph a min-cost spanning tree?