"Shift Homomorphism"

Consider the following fragment of HPF pseudocode:

```
!HPF$  TEMPLATE tx (0 :  ), ty (0 :  )
!HPF$  DISTRIBUTE tx (BLOCK (w_x)) ONTO p
!HPF$  DISTRIBUTE ty (BLOCK (w_y)) ONTO p

!HPF$  DIMENSION x (0 :  n - 1), y (0 :  n - 1)
!HPF$  ALIGN x (i) WITH tx (k_x+ s_tx* i)
!HPF$  ALIGN y (i) WITH ty (k_y+ s_ty* i)

       x = y
```

By suitable composition of section and alignment parameters, possibly combined with trivial relabelling so that index ranges start at 0, any assignment between one-dimensional sections block-distributed over the same one-dimensional processor arrangement can be reduced to this case.

The arrays `x` and `y` are "shift-homomorphic" iff

$$s_{tx}/w_x = s_{ty}/w_y$$

If this condition holds the assignment of `y` to `x` can be implemented by shifting the values of `y` along template `ty` then performing a local copy.

Proof:

The element i of `x` is aligned with element $k_x + s_{tx}i$ of template `tx` and the element i of `y` is aligned with element $k_y + s_{ty}i$ of template `ty`.

A temporary array `y'` is created by shifting the values in `y` by amount

$$a = k'_y - k_y$$

along template `ty`. So element i of `y'` is aligned with element $k'_y + s_{ty}i$ of template `ty`.

In particular, choose

$$k'_y = \lfloor k_x w_y / w_x \rfloor$$

so the alignment of element i of `y'` is to element

$$\begin{aligned} k'_y + s_{ty}i &= \lfloor k_x w_y/w_x \rfloor + s_{ty}i \\ &= \lfloor k_x w_y/w_x + s_{ty}i \rfloor \\ &= \lfloor (k_x w_y + s_{ty} w_x i)/w_x \rfloor \end{aligned}$$

of template `ty`.

Now, element j_x of template `tx` is mapped to processor $\lfloor j_x/w_x \rfloor$ and element j_y of template `ty` is mapped to processor $\lfloor j_y/w_y \rfloor$.

So element i of `x` is mapped to processor

$$\lfloor (k_x + s_{tx} i)/w_x \rfloor$$

and element i of `y'` is mapped to processor

$$\lfloor (k'_y + s_{ty} i)/w_y \rfloor$$

But

$$\begin{aligned}
\lfloor (k'_y + s_{ty} i)/w_y \rfloor &= \lfloor \lfloor (k_x w_y + s_{ty} w_x i)/w_x \rfloor / w_y \rfloor \\
&= \lfloor (k_x w_y + s_{ty} w_x i)/w_x w_y \rfloor \\
&= \lfloor k_x/w_x + s_{ty} i/w_y \rfloor \\
&= \lfloor (k_x + s_{tx} i)/w_x \rfloor
\end{aligned}$$

In the second step we used a general property of non-negative integers:

$$\lfloor \lfloor l/m \rfloor / n \rfloor = \lfloor l/mn \rfloor$$

and in the last step we used the shift homomorphism condition.

This establishes that elements i of `x` and `y'` map to the same processor. Suppose this is processor n. Assuming array segments are allocated according to the template size, the local subscript in `x` is

$$l_x = k_x + s_{tx} i - n * w_x$$

and the local subscript in `y'` is

$$l'_y = k'_y + s_{ty} i - n * w_y$$

Using the shift homomorphism condition gives

$$w_x l'_y - w_y l_x = w_x k'_y - w_y k_x$$

or

$$l'_y = (w_y/w_x) l_x + (k'_y - w_y k_x/w_x)$$

as the relation between the local subscripts.